
Heavy Quarkonium*

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Abstract

Bound states of heavy $\bar{q}q$ quarks are reviewed within the context of QCD. First of all, we consider the calculations which can be performed *ab initio*, which includes $t\bar{t}$ with principal quantum number n up to four, $b\bar{b}$ states with $n = 1$ and (to a lesser extent) $\bar{b}b$ with $n = 2$ and $\bar{c}c$ for $n = 1$. Among the results, we report a very precise $O(\alpha_s^4)$ evaluation of b, c quark masses from quarkonium spectrum with a potential to two loops, a calculation of the decay $\Upsilon \rightarrow e^+e^-$ and a prediction for the splitting $\Upsilon - \eta_b$. We then consider how, with the help of reasonable assumptions, one can extend QCD calculations to other states of heavy quarks. Finally, a few words are said on the treatment of light quark bound states.

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1. Introduction

In the present lectures we are going to review some aspects of the analysis of heavy quarkonia, $\bar{t}t$, $\bar{c}c$ and especially $\bar{b}b$ states. Before the advent of asymptotic freedom in 1973, hadronic interactions were analyzed with (among other methods) the help of the *quark model*, which incorporated a somewhat inconsistent set of semiphenomenological rules. An important rôle was played by bound state calculations in the so-called *constituent quark model*, developed in the early sixties by, among others, Morpurgo, Dalitz and collaborators, and Oliver, Pene, Reynal and Le Yaouanc. In this model u, d, s quarks were given phenomenological masses of 300 – 500 MeV, and were bound by potentials: the harmonic oscillator potential being a popular choice because of its simplicity. Quite surprisingly, a large number of properties of hadrons could be reproduced in this way.

After the advent of asymptotic freedom, and with it of a consistent field theory of strong interactions, it was possible to reformulate the quark model in terms of QCD. Thus, De Rújula, Georgi and Glashow^[1] showed that taking into account relativistic corrections and colour algebra one could calculate the spectrum of the then known hadrons, including in particular such properties as the $N - \Delta$ mass difference, and even the $\Sigma^0 - \Lambda$ splitting, something that had defied previous, non-QCD analyses. They were also able to *predict* qualitative features of the charmonium spectrum.

Nowadays we expect more from QCD, at least for heavy quarks. The reason is that there, and to leading order in $\langle v^2 \rangle$ ($\langle v^2 \rangle$ the average velocity of the quarks), the interaction is equivalent to a *potential*. At very short distances this potential has to be of the coulombic type,

$$-\frac{C_F \alpha_s}{r}. \quad (1.1)$$

Even in the static limit, in QCD, (1.1) will be modified by radiative corrections; but these should be of the form of a function of r . In fact, also at long distances one expects, in the nonrelativistic (NR) limit, a local potential with the form of a function $U(r)$, although it will not be of coulombic type. The reason for this is galilean invariance, that will hold in the NR limit. By virtue of it, we must have that the derivative of the position \mathbf{Q} , the velocity, should be proportional to the momentum:

$$[H, \mathbf{Q}] = \frac{i}{\hbar} \dot{\mathbf{Q}} = \frac{i}{\hbar m} \mathbf{P},$$

with H the hamiltonian. If we define the interaction by $H_{\text{int}} \equiv H - \mathbf{P}^2/2m$ and evaluate the commutator above, it follows that $[H_{\text{int}}, \mathbf{Q}] = 0$ and so, because of a well-known theorem of von Neumann, H_{int} must be a function of \mathbf{Q} . A function which due to rotational invariance, and at least neglecting spin, may only depend on r :

$$H_{\text{int}} = U(r).$$

Needless to say, *relativistic* corrections will in general not be representable by local, momentum-independent potentials, as is the case even in QED. In QCD one encounters QED-like corrections and idiosyncratic QCD ones, in particular those associated with the complicated structure of the vacuum. Of these the most important are the effects involving the gluon condensate $\langle \alpha_s : G^2 : \rangle$, first studied in this context by Leutwyler and Voloshin^[2]; the quark condensate also gives contributions but, for heavy quarkonium, subleading ones.

We will consider in these lectures bound states of heavy quarks (and at the end say a few words about light quarks¹) in decreasing order of tractability by rigorous QCD. First of all, we will consider situations where one has the inequalities

$$a \ll R; \quad |B_n| \gg \Lambda; \quad m \gg \Lambda$$

where a is the equivalent of the Bohr radius, and B_n are the equivalent of the Balmer energies; m is the quark mass, and $\Lambda \simeq 300$ MeV is the QCD parameter. Under these circumstances, nonperturbative (NP) and confinement effects are expected to be small, and so are the radiative and relativistic corrections. All of them may therefore be treated as *perturbations* of a nonrelativistic, unconfined and leading order (in the QCD coupling α_s) system. This will constitute the bulk of these lectures. Then we will diverge from

¹ Light quark bound states, and gluonic bound states, are reviewed in the companion lectures by Yu. Simonov.

the situations where one can effect *ab initio* QCD calculations, relying more and more on reasonable (but unproven) assumptions, and on models.

2. Heavy quarks at short distances: pure QCD analysis

For very heavy $\bar{q}q$ bound states the equivalent of the Bohr radius, $a = 2/(mC_F\alpha_s)$, is much smaller than the confinement radius, $R \sim \Lambda^{-1}$. So we expect that, for lowest n states, with n the principal quantum number, confinement may be neglected, or at least treated as a first order perturbation. In this case the quarkonium system is very similar to a familiar one, viz., positronium; so that methods analogous to those developed for the last may be applied also to the study of the former. The NR potential may be obtained from *perturbative* QCD; note that, unlike for positronium, and because of the zero mass of the gluons, radiative corrections are present even in the static limit. The leading piece is given by the tree level nonrelativistic amplitude and we then include higher effects as perturbations: we proceed in steps.

We will also follow the method of *equivalent potentials*, advocated for QCD by Gupta and collaborators^[3]. We find this method the more transparent one. Other, equivalent methods, based on the Bethe–Salpeter equation or effective lagrangians may be found in the literature^[4].

In the method of equivalent potentials one profits from the fact that, in the NR limit, the potential is given by the Fourier transform of the scattering amplitude, in the Born approximation:

$$T_{NR}^{\text{Born}}(\mathbf{p} \rightarrow \mathbf{p}') = -\frac{1}{4\pi^2} \int d^3\mathbf{r} e^{i\mathbf{r}(\mathbf{p}-\mathbf{p}')} V(r), \quad (2.1)$$

where \mathbf{p}, \mathbf{p}' are the initial and final momenta in the center of mass reference system. On the other hand, T_{NR}^{Born} may be calculated as the nonrelativistic limit of the relativistic scattering amplitude:

$$T_{NR}^{\text{Born}} = \lim_{m \rightarrow \infty} \frac{1}{4\sqrt{p_{10}p_{20}p'_{10}p'_{20}}} F(p_1 + p_2 \rightarrow p'_1 + p'_2);$$

formally the NR limit is equivalent to taking the limit of infinite quark masses, keeping the three-momenta fixed. One can thus calculate the Born approximation to F , F^{Born} using the familiar Feynman rules (actually at tree level) for $\bar{q}q$ scattering, take the NR limit and hence obtain T_{NR}^{Born} . From it, by inverting (2.1) one finds V . What is more, we can calculate corrections to V by including corrections (in particular, relativistic) to T_{NR}^{Born} .

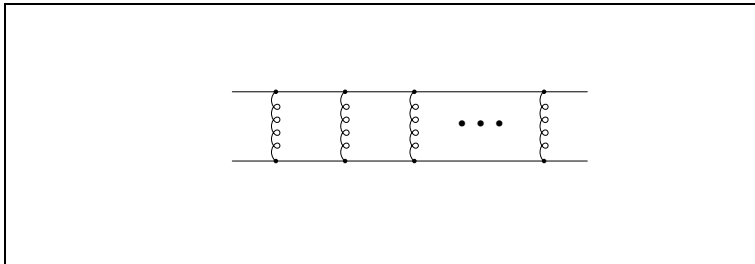


FIGURE 1A. A ladder of gluon exchanges.

A few words will now be said about the connection of the method with Feynman graph calculations; more details may be found in Ch. 6 of ref. 5, which we roughly follow in these notes. Working in the strict nonrelativistic limit to avoid inessential complications, it is easy to check that solving the Schrödinger equation with the coulombic potential of Eq. (1.1) is equivalent to summing an infinite ladder of exchange graphs (Fig. 1A): indeed, both methods yield the same S matrix. Including radiative corrections then alters the potential, as will be shown below. The corresponding Schrödinger equation will be equivalent to dressing the kernel F_N associated with the calculated radiative corrections to N loops with ladders, corresponding to the coulombic wave functions, as shown graphically in Fig. 1B where the kernel would be given, e.g. to two loops, by graphs depicted in Figs. 2 3 below. Thus the method of potentials is equivalent to a particular arrangement of the summation of perturbation theory.

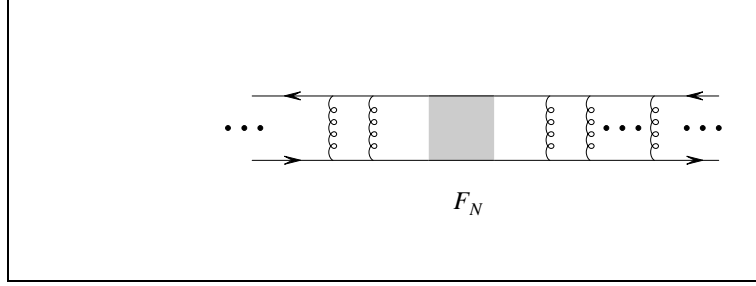


FIGURE 1B. Kernel, dressed with infinite sums of ladders.

2.1. Step 1: coulombic Schrödinger equation

According to what we have said, we proceed in steps. In this first step we consider a nonrelativistic tree-level kernel (Fig. 2). Here the scattering amplitude is such that it generates a coulombic potential. So we get the Schrödinger equation, for the energies and wave functions,

$$H^{(0)}\Psi_n^{(0)} = E_n^{(0)}\Psi_n^{(0)}$$

$$H^{(0)} = 2m - \frac{1}{m}\Delta + V^{(0)}(r), \quad V^{(0)}(r) = -\frac{C_F\alpha_s}{r};$$

note that for quarkonium the reduced mass is $m/2$. The quantities m , α_s however are as yet undefined; only when including radiative corrections will we be able to give them a precise meaning. This equation, formally identical to that of an hydrogen-like atom, can be solved exactly, and will be our starting point in the calculations.

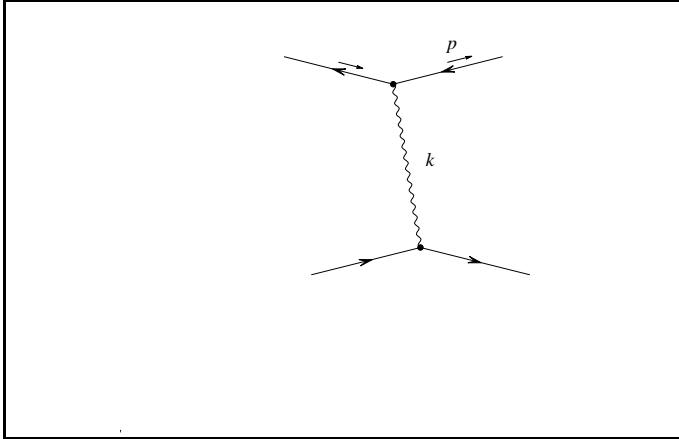


FIGURE 2. One-gluon exchange.

2.2. Step 2: relativistic corrections

The relativistic corrections are identical to those for positronium, known since ancient times. They are found by considering still the tree level scattering amplitude corresponding to the diagram of Fig. 2, but keeping now the terms of order $1/c^2$, c the speed of light; the details of the derivation may be found in textbooks on relativistic quantum mechanics^[6]. Adding also the correction to the kinetic energies,

$$\sqrt{m^2 - \Delta} \simeq m - \Delta/2m - \Delta^2/8m^3$$

one finds the hamiltonian,

$$H_{\text{tree}} = H^{(0)} + V_{\text{rel}}^{(0)} \quad (2.2a)$$

where the superscript zero in $V^{(0)}$ indicates that the potential is still obtained from a tree level (zero loop) amplitude. The relativistic corrections, which are to be treated as first order perturbations to the unperturbed equation (2.1), read

$$V_{\text{rel}}^{(0)} = V_{\text{si;rel}}^{(0)} + V_{\text{tens}}^{(0)} + V_{LS}^{(0)} + V_{\text{hf}}^{(0)}. \quad (2.2b)$$

The various pieces, spin-independent (in which we also include the correction to the kinetic energy), tensor, LS and hyperfine, are

$$\begin{aligned} V_{\text{si; rel}}^{(0)} &= -\frac{1}{4m^3}\Delta^2 + \frac{C_F\alpha_s}{m^2}\frac{1}{r}\Delta, \\ V_{\text{tens}}^{(0)} &= \frac{C_F\alpha_s}{4m^2}\frac{1}{r^3}S_{12}, \\ V_{LS}^{(0)} &= \frac{3C_F\alpha_s}{2m^2}\frac{1}{r^3}\mathbf{L}\mathbf{S}, \\ V_{\text{hf}}^{(0)} &= \frac{4\pi C_F\alpha_s}{3m^2}\mathbf{S}^2\delta(\mathbf{r}). \end{aligned} \tag{2.2c}$$

Here \mathbf{L} is the orbital angular momentum operator, \mathbf{S} the total spin operator, and S_{12} the tensor operator:

$$\mathbf{L} = -i\mathbf{r} \times \nabla, \quad \mathbf{S} = \frac{\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2}{2}, \quad S_{12} = 2 \sum_{ij} \left(\frac{2r_i r_j}{r^2} - \delta_{ij} \right) S_i S_j.$$

The wave functions are assumed to have spinor components, and the Pauli matrices $\boldsymbol{\sigma}_a$ act on spinor $\chi(\lambda_a)$, $a = 1, 2$.

2.3. Step 3: radiative corrections

Before discussing the radiative corrections, a matter has to be settled first, which is that of the meaning of the mass in the Schrödinger equation. We have defined the potential by assuming that it vanishes at infinity; otherwise, we have the ambiguity of an arbitrary constant. Thus, we must interpret the mass as the mass at long distances, i. e., on the mass shell. Now, both these requirements are not rigorously valid since quarks are confined in a region of radius $R \sim \Lambda^{-1}$; but we can work with them to the extent that R is much larger than the region where the movement of the quarks takes place, $a = 2/mC_F\alpha_s$.

With this requirement we define m , called the mass shell, or pole mass, to be such that, *in perturbation theory*,

$$S_{\text{p.t.}}(\not{p} = m)^{-1} = 0. \tag{2.3}$$

The relation of this mass with the $\overline{\text{MS}}$ mass, \bar{m} was found by Coquereaux and Tarrach^[7] to one loop and by Gray et al.^[8] to two loops. After correcting a misprint of the last reference one finds

$$m = \bar{m}(\bar{m}^2) \left\{ 1 + \frac{C_F\alpha_s(\bar{m}^2)}{\pi} + (K - 2C_F) \left[\frac{\alpha_s(\bar{m}^2)}{\pi} \right]^2 \right\}, \tag{2.4a}$$

where, denoting by n_f the number of quark flavours with mass less than or equal to m ,

$$\begin{aligned} K &= \frac{1}{9}\pi^2 \log 2 + \frac{7}{18}\pi^2 - \frac{1}{6}\zeta(3) + \frac{3673}{288} - \left(\frac{1}{18}\pi^2 + \frac{71}{144} \right) n_f + \sum_{i=1}^{n_f-1} \Delta\left(\frac{m_i}{m}\right), \\ \Delta(\rho) &= \frac{4}{3} \left\{ \frac{1}{8}\pi^2 \rho - \frac{3}{4}\rho^2 + \dots \right\}. \end{aligned} \tag{2.4b}$$

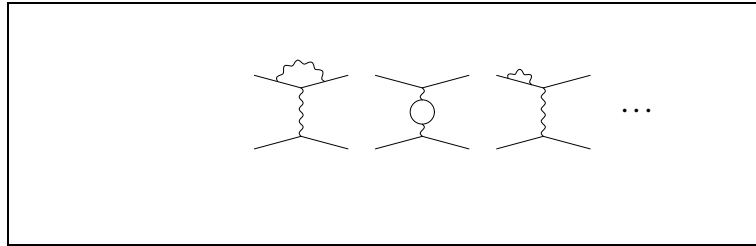


FIGURE 3. Some radiative corrections.

All other quantities, however, are renormalized in the $\overline{\text{MS}}$ scheme. So, $\alpha_s(\mu^2)$ will be the $\overline{\text{MS}}$ coupling at the scale μ^2 , etc. These radiative corrections (some of which are shown in Fig. 3) have been evaluated by a number of people. Those to the spin-independent part of the potential, in the strict static approximation, were first calculated by Fischler and Billoire^[9] to one loop and by Peter and Schröder^[10] (who checked and

corrected a mistake of Peter's calculation) to two loops. Because of the zero mass of the gluons, corrections nonanalytic in the average velocity occur. These depend on $|v|$ and are of a size comparable to that of the two loop static corrections; note that, in the coulombic approximation, $\langle |v| \rangle = C_F \alpha_s$. They were calculated, together with $O(v^2)$ radiative corrections to one loop, in ref. 11. Spin-dependent one loop corrections were evaluated in refs. 3, 12.

We discuss in some detail the spin-independent part of the spectrum. To take into account all terms giving corrections of $O(\alpha_s^4)$ to the energy spectrum one writes the Hamiltonian as

$$H = \tilde{H}^{(0)} + H_1 \quad (2.5a)$$

where $\tilde{H}^{(0)}$ may, and will, be solved exactly and contains all the coulombic pieces of the interaction:

$$\begin{aligned} \tilde{H}^{(0)} = & 2m + \frac{-1}{m} \Delta - \frac{C_F \tilde{\alpha}_s(\mu^2)}{r}; \\ \tilde{\alpha}_s(\mu^2) = & \alpha_s(\mu^2) \left\{ 1 + \left(a_1 + \frac{\gamma_E \beta_0}{2} \right) \frac{\alpha_s(\mu^2)}{\pi} \right. \\ & \left. + \left[\gamma_E \left(a_1 \beta_0 + \frac{\beta_1}{8} \right) + \left(\frac{\pi^2}{12} + \gamma_E^2 \right) \frac{\beta_0}{4} + a_2 \right] \frac{\alpha_s}{\pi^2} \right\}. \end{aligned} \quad (2.5b)$$

H_1 is to be considered as a *perturbation*. Its form is,

$$H_1 = V_{\text{s.i.;rel}} + V_1^{(L)} + V_2^{(L)} + V^{(LL)} + V_{\text{s.rel}} + V_{\text{hf}}, \quad (2.5c)$$

$$\begin{aligned} V_{\text{s.i.;rel}} = & \frac{-1}{4m^3} \Delta^2 + \frac{C_F \alpha_s}{m^2 r} \Delta, \\ V_1^{(L)} = & \frac{-C_F \alpha_s(\mu^2)^2}{\pi} \frac{\beta_0 \log r \mu}{2 r}, \\ V_2^{(L)} = & \frac{-C_F \alpha_s^3}{\pi^2} \left(a_1 \beta_0 + \frac{\beta_1}{8} + \frac{\gamma_E \beta_0^2}{2} \right) \frac{\log r \mu}{r}, \\ V^{(LL)} = & \frac{-C_F \beta_0^2 \alpha_s^3}{4\pi^2} \frac{\log^2 r \mu}{r}, \\ V_{\text{s.rel}} = & \frac{C_F b_1 \alpha_s^2}{2m r^2}, \\ V_{\text{hf}} = & \frac{4\pi C_F \alpha_s}{3m^2} s(s+1) \delta(\mathbf{r}). \end{aligned} \quad (2.5d)$$

$V_{\text{s.i.;rel}}$ is the spin-independent piece of V_{rel} in Eq. (2.2c); $V_{\text{s.rel}}$ is a one loop velocity-dependent correction and all other ones are one and two loop static corrections with the exception of the last term, representing hyperfine splitting. Although we are considering the spin-independent interaction, to the precision we are working one needs to differentiate between the masses of vector and pseudoscalar states, hence the presence of this piece. a_1 was calculated in ref. 9, a_2 in ref. 10 and b_1 and many of the rest of the terms in ref. 11; all these constants are given in the Appendix. Note that, to the precision we are working, α_s has to be evaluated to three loops

$$\begin{aligned} \alpha_s(Q^2) = & \frac{4\pi}{\beta_0 L} \left\{ 1 - \frac{\beta_1 \log L}{\beta_0^2 L} + \frac{\beta_1^2 \log^2 L - \beta_1^2 \log L + \beta_2 \beta_0 - \beta_1^2}{\beta_0^4 L^2} \right\}, \\ L = & \log Q^2 / \Lambda^2. \end{aligned}$$

Some bookkeeping is necessary to identify which terms to include for a given order of accuracy in the calculation, e.g. in the evaluation of the energy levels. The pieces given here will provide a calculation accurate to order α_s^4 . All terms in H_1 are to be treated as first order perturbations of $\tilde{H}^{(0)}$, except for the term $V_1^{(L)}$, which has to be evaluated to second order. Thus it produces, in addition to the first order contribution,

$$\delta_{V_1^{(L)}}^{(1)} E_{10} = -m \frac{\beta_0 C_F^2 \alpha_s^2(\mu^2) \tilde{\alpha}_s(\mu^2)}{4\pi} \left(\log \frac{a}{2} + 1 - \gamma_E \right), \quad (2.6a)$$

the second-order energy shift, for the ground state,^[13]

$$\delta_{V_1^{(L)}}^{(2)} E_{10} = -m \frac{\beta_0 C_F^2 \alpha_s^4}{4\pi^2} \left[N_0 \log^2 \frac{a\mu}{2} + N_1 \log \frac{a\mu}{2} + N_2 \right]; \quad (2.6b)$$

the constants N are given in the Appendix.

The first order contributions of the other V 's are easily evaluated using the formulas of ref. 11. One finds

$$E_{nl}^{\text{p.t.}} = 2m - m \frac{C_F^2 \tilde{\alpha}_s^2}{4n^2} + \sum_V \delta_V^{(1)} E_{nl} + \delta_{V_1^{(L)}}^{(2)} E_{nl}. \quad (2.7)$$

The label “p.t.” in $E_{nl}^{\text{p.t.}}$ indicates that we have as yet only used results deduced from perturbation theory; the full expression would be

$$E_{nl} = E_{nl}^{\text{p.t.}} + \delta_{\text{NP}} E_{nl},$$

with $\delta_{\text{NP}} E_{nl}$ given below, Eq. (2.10).

The $\delta_V^{(1)} E_{nl}$ are, with a as above,

$$\begin{aligned} \delta_{V_{\text{tree}}}^{(1)} E_{nl} &= -\frac{2}{n^3 m^3 a^4} \left[\frac{1}{2l+1} - \frac{3}{8n} \right] + \frac{C_F \alpha_s}{m^2} \frac{2l+1-4n}{n^4 (2l+1) a^3}; \\ \delta_{V_2^{(L)}}^{(1)} E_{nl} &= -\frac{C_F c_2^{(L)} \alpha_s^3}{\pi^2 n^2 a} \left[\log \frac{na\mu}{2} + \psi(n+l+1) \right]; \\ \delta_{V_{(LL)}}^{(1)} E_{nl} &= -\frac{C_F \beta_0^2 \alpha_s^3}{4\pi^2 n^2 a} \left\{ \log^2 \frac{na\mu}{2} + 2\psi(n+l+1) \log \frac{na\mu}{2} \right. \\ &\quad \left. + \psi(n+l+1)^2 + \psi'(n+l+1) \right. \\ &\quad \left. + \theta(n-l-2) \frac{2\Gamma(n-l)}{\Gamma(n+l+1)} \sum_{j=0}^{n-l-2} \frac{\Gamma(2l+2+j)}{j!(n-l-j-1)^2} \right\}; \\ \delta_{V_{\text{s,rel}}}^{(1)} E_{nl} &= \frac{C_F b_1 \alpha_s^2}{m} \frac{1}{n^3 (2l+1) a^2}. \end{aligned} \quad (2.8a)$$

Here and from now on we have defined $a = 2/(C_F m \tilde{\alpha}_s)$. We recall that constants are collected in the Appendix. For the masses of the vector states (Υ , Υ' , Υ'' ; J/ψ , ψ' , ...) one has to add the hyperfine shift, at tree level,

$$\delta_{V_{\text{spin}}}^{(1)} E_{nl} = \delta_{s1} \delta_{l0} \frac{8C_F \alpha_s}{3n^3 m^2 a^3}. \quad (2.8b)$$

The value of the contributions of $V_1^{(L)}$ were given above, Eq. (2.6).

2.4. Step 4: nonperturbative corrections

The *leading* nonperturbative (NP) corrections can be shown to be those associated with the contribution of the gluon condensate. Physically they may be understood as follows. We consider that the quarks move in a medium, the QCD vacuum, which is full of soft gluons (Fig. 4) that we represent by their field strength operators, $G_{\mu\nu}^c(x)$. When $a \ll R$, we may assume that the confinement size is infinite and, moreover, that one can neglect the fluctuations of the $G_{\mu\nu}^c(x)$ in the region of size a in which the quarks move. So we approximate the effect of the motion in the gluon soup by introducing an interaction, which in the static limit will be of dipole type, of the quarks with a constant gluonic field:

$$H_{NP} = -g r_i G_{0i}^c(0) t^c = -g \mathbf{r} \mathbf{E}^c t^c.$$

Because of Lorentz invariance of the vacuum we assume that $\langle G_{\mu\nu}^c \rangle = 0$, but $\langle \alpha_s : G^2 : \rangle \neq 0$. For dimensional reasons, this will give the leading NP contribution to the spin-independent energy shifts. Applying thus

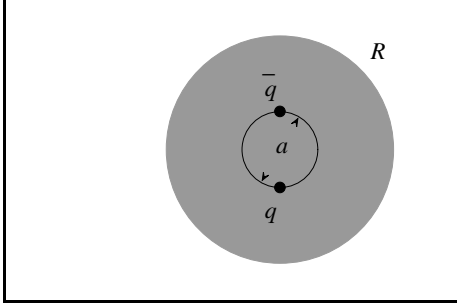


FIGURE 4. The region where the $\bar{q}q$ pair move inside the confinement region.

straightforward second order perturbation theory we have²

$$\delta_{\text{NP}} E_{nl} = - \left\langle \Psi_{nlM}^{(0)}, H_{\mathcal{E}} \frac{1}{H^{(8)} - E_n^{(0)}} H_{\mathcal{E}} \Psi_{nlM}^{(0)} \right\rangle.$$

There are a few points to clarify regarding this equation. First of all, we may, since we average over directions, neglect the magnetic quantum number M . Secondly, we have used in the denominator the *octet* Hamiltonian,

$$H^{(8)} = -\frac{1}{m}\Delta + \frac{1}{2N_c} \frac{\alpha_s}{r}.$$

This happens because the perturbed state, $H_{\mathcal{E}}|\Psi^{(0)}\rangle = -g\mathbf{r}\mathcal{E}_a t^a |\Psi^{(0)}\rangle$, is manifestly an octet one, as \mathcal{E}_a creates a gluon on top of the singlet $|\Psi^{(0)}\rangle$.

Next, we have to relate expectation values of products $\mathcal{E} \dots \mathcal{E}$ to the gluon condensate. For this, first write the gluon radiation Hamiltonian as

$$H_{\text{rad}} = \frac{1}{8\pi} \int d^3\mathbf{r} : \mathcal{E}^2 + \mathcal{B}^2 :,$$

with sums over omitted colour indices understood. Its expectation value in the physical vacuum should vanish, so we conclude

$$\langle \text{vac} | \int d^3\mathbf{r} : \mathcal{E}^2 : | \text{vac} \rangle = - \langle \text{vac} | \int d^3\mathbf{r} : \mathcal{B}^2 : | \text{vac} \rangle.$$

We assume the field intensities to be constant, so we may replace the integrals by the volume times the integrands at $x = 0$. Canceling then the volume and recalling that $G^2 = -2(\mathcal{E}^2 - \mathcal{B}^2)$, we find

$$\langle \text{vac} | : \mathcal{E}^2 : | \text{vac} \rangle = -\frac{1}{4} \langle \text{vac} | : G^2(0) : | \text{vac} \rangle.$$

Finally, using Lorentz and colour invariance of the physical vacuum,

$$g^2 \langle : \mathcal{E}_a^i(0) \mathcal{E}_b^j(0) : \rangle = \frac{4\pi\alpha_s \delta_{ij} \delta_{ab}}{(D-1)(N_c^2-1)} \langle : \mathcal{E} \mathcal{E} : \rangle = -\frac{\pi \delta_{ij} \delta_{ab}}{24} \langle \alpha_s G^2 \rangle.$$

With this we get

$$\begin{aligned} \delta_{\text{NP}} E_{nl} &= \left\langle \Psi_{nl}, \left(\mathbf{r} \sum_a t^a \mathcal{E}_a(0) \right) \frac{1}{H^{(8)} - E_n^{(0)}} H_{\mathcal{E}} \left(\mathbf{r} \sum_b t^b \mathcal{E}_b(0) \right) \Psi_{nl} \right\rangle \\ &= \frac{\pi \langle \alpha_s G^2 \rangle}{18} \sum_i \left\langle \Psi_{nl}, r_i \left(-\frac{1}{m}\Delta + \frac{\alpha_s}{6r} - E_n^{(0)} \right)^{-1} r_i \Psi_{nl} \right\rangle. \end{aligned} \quad (2.9)$$

To finish the calculation we have to invert the operator $(-m^{-1}\Delta + \alpha_s/6r)$. For the simple case above, the method may be found in ref. 2; in more complicated situations, cf. ref. 14. So we finally obtain the nonperturbative pieces of the energy shifts, which are of the form^[2],

$$\delta_{\text{NP}} E_{nl} = m \frac{\pi n^6 \epsilon_{nl} \langle \alpha_s : G^2 : \rangle}{(m C_F \alpha_s)^4}, \quad (2.10)$$

² Projectors over the subspace orthogonal to $|\Psi^{(0)}\rangle$, that we do not write explicitly, are understood.

where the numbers ϵ_{nl} are of order unity, $\epsilon_{10} \simeq 1.5$. The evaluations for the spin-dependent shifts may be found in ref. 14 (with a minor correction in ref. 15) and the contributions of *higher* order operators has been considered in refs. 16, 17. Note that, as already remarked by Leutwyler^[2], one cannot derive (2.10) from a local potential; but, for the lowest states, the effect may be *approximated* by a cubic one,

$$V_{\text{Gluon cond.}}(r) \sim \Lambda^4 r^3.$$

As promised, the correction (2.10) is relativistic in that it is of order $1/m^4$; but the coefficient is very large because of the high powers of n , α_s^{-1} . The reason for these powers of α_s and n can be understood easily. Two come from the energy denominators, and four from the expectation value $\langle r_i r_j \rangle_{nl} \sim n^4 / (m C_F \alpha_s)^2$.

The right hand side of (2.10) grows as the sixth power of the radial quantum number, n . It is in fact this very fast growth with n that leads to the breakdown of the method as soon as n exceeds, or in some cases even equals, the value 2 for $\bar{c}c$, $\bar{b}b$; only for $\bar{t}t$ can one go to $n \sim 5$.

The NP corrections to the wave function may be obtained with methods similar to those employed to evaluate $\delta_{\text{NP}} E$. For $n = 1$, $l = 0$, we have

$$\Psi_{10}(r) \rightarrow (1 + \delta_{\text{NP}}(r)) \Psi_{10}(r), \quad (2.11a)$$

where the NP correction is

$$\delta_{\text{NP}}(r) = \left\{ \frac{2968}{425} - \frac{104}{425} \rho^2 - \frac{52}{1275} \rho^3 - \frac{1}{225} \rho^4 \right\} \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 (C_F \alpha_s)^6}, \quad \rho = \frac{2r}{a}. \quad (2.11b)$$

It turns out that the coefficient of the correction is larger than for the energy shifts, both in powers of α_s^{-1} and of n : the effects of confinement are larger for the wave function than for the energy levels.

The contribution of some higher dimensional operators has been estimated by Pineda^[16]; the corrections due to the finite size of the hadron is discussed in ref. 17, and will be briefly reviewed later.

Radiative and nonperturbative corrections to higher excited states may likewise be evaluated; as can be calculated the decay rates into photons or leptons, e.g., $\mathcal{T} \rightarrow e^+ e^-$, $\eta_b \rightarrow \gamma \gamma$. We will present some of these results below.

3. Results

Let us summarize the results. The calculation is fully justified, in the sense that higher order corrections (both perturbative and NP) are smaller than lower order ones for $\bar{b}b$ with $n = 1$. The same is partially true for the energy levels of the same states with $n = 2$ and, for $\bar{c}c$, for $n = 1$. For the wave functions of $\bar{b}b$, $n \geq 2$ and all $\bar{c}c$ states, and for the energy levels with higher values of n than the ones reported above, the calculation is meaningless as nominally subleading corrections overwhelm nominally leading ones.

Before presenting the results a few words have to be said about the choice of the renormalization point, μ . As our Eqs. (2.6), (2.8) show, a *natural* value for this parameter is

$$\mu_0 = \frac{2}{na} = \frac{m C_F \tilde{\alpha}_s}{n},$$

for states with the principal quantum number n , and this will be our choice. For states with $n = 1$ the results of the calculation will turn out to depend little on the value of μ , provided it is reasonably close to μ_0 . Higher states are another matter; we will discuss our choices when we consider them.

As input parameters we take the recent determinations^[18],

$$\Lambda(n_f = 4, \text{ three loops}) = 0.283 \pm 0.035 \text{ GeV} \quad [\alpha_s(M_Z^2) \simeq 0.117 \pm 0.024],$$

and for the gluon condensate, very poorly known, the value

$$\langle \alpha_s G^2 \rangle = 0.06 \pm 0.02 \text{ GeV}^4.$$

(Note that the slight difference between the results reported below and those of previous determinations^[11,13] are mostly due to the variation of the preferred value of Λ from 0.20 to 0.28 GeV.)

3.1. $n = 1$ states

For $\bar{b}b$ one gets a precise determination of m_b , and less so of $\bar{m}_b(\bar{m}_b^2)$ (pole and $\overline{\text{MS}}$ masses), a reliable prediction for the hyperfine splitting, and reasonable agreement with the experimental value of $\Upsilon \rightarrow e^+e^-$; these will be discussed later. For $\bar{c}c$ a reasonably accurate value is also obtained for m_c . For $\bar{b}b$, and with Λ , $\langle\alpha_s G^2\rangle$ as given before and varying μ^2 around μ_0^2 by 25% to estimate the systematic errors of the calculation one finds, from the Υ mass, the quark masses^[13]:

$$\begin{aligned} m_b &= 5.065 \pm 0.043 (\Lambda) \mp 0.005 (\langle\alpha_s G^2\rangle) \\ &\quad {}_{+0.037}^{-0.031} (\text{vary } \mu^2 \text{ by } 25\%) \pm 0.006 (\text{other th. uncert.}) \\ \bar{m}_b(\bar{m}_b^2) &= 4.455 \pm 0.012 (\Lambda) \mp 0.005 (\langle\alpha_s G^2\rangle) \\ &\quad {}_{+0.034}^{-0.029} (\text{vary } \mu^2 \text{ by } 25\%) \pm 0.006 (\text{other th. uncert.}). \end{aligned} \quad (3.1)$$

Note that m_b is correct to $O(\alpha_s^4(\mu_0^2))$, and $\bar{m}_b(\bar{m}_b^2)$ to $O(\alpha_s^2(m_b^2))$. The piece denoted by the expression “other th. uncert.” in (3.1) refers to the error coming from higher dimensional operators and some higher order perturbative terms; it can be found discussed in refs. 13, 16. It is comfortably smaller than the errors due to the uncertainty on Λ , $\langle\alpha_s G^2\rangle$.

The values of μ_0^2 , $\alpha_s(\mu_0^2)$, $\tilde{\alpha}_s(\mu_0^2)$ are, respectively,

$$\mu_0^2 = 7.86 \text{ GeV}^2, \alpha_s(\mu_0^2) = 0.257, \tilde{\alpha}_s(\mu_0^2) = 0.415.$$

We see that α_s is small, thus justifying the use of perturbation theory. Moreover, $a/2 \simeq (2.8 \text{ GeV})^{-1} \ll \Lambda^{-1}$: the quarks move well away from the confinement region. Finally, the binding energy is also considerably larger than Λ . Thus we find our approximations justified *a posteriori*.

For $\bar{c}c$, and with $\Lambda(n_f = 3, \text{ three loops}) = 0.338 \pm 0.037$ and the mass of the J/ψ as an input now,

$$\begin{aligned} m_c &= 1.936 {}_{-0.068}^{+0.059} (\Lambda) \mp 0.014 (\langle\alpha_s G^2\rangle) \\ &\quad {}_{+0.106}^{-0.124} (\text{varying } \mu^2 \text{ by } 25\%) \pm 0.014 (\text{th. uncert.}) \\ \bar{m}_c(\bar{m}_c^2) &= 1.564 {}_{-0.035}^{+0.086} (\Lambda) \mp 0.013 (\langle\alpha_s G^2\rangle) \\ &\quad {}_{+0.119}^{-0.095} (\text{varying } \mu^2 \text{ by } 25\%) \pm 0.013 (\text{th. uncert.}), \end{aligned} \quad (3.2)$$

and $\mu_0^2 = 2.871 \text{ GeV}^2$ now. The errors, and the values of α_s , $\tilde{\alpha}_s$ increase correspondingly and it follows that the calculation is much less reliable than for the $\bar{b}b$ case, as the errors in (3.2) show.

The values of the b quark masses reported here, e.g., Eq. (3.1), are slightly larger than those one finds with the sum rule method (see for example, refs. 19). It is not clear to the author why this occurs. I suspect that the sum rule evaluations contain systematic uncertainties which are not under control; and indeed, the determinations are not very compatible one with another. Anyway, the discrepancies are not terribly large.

3.2. $n = 2$ states

For the states with $n = 2$, the energy levels can be evaluated using the values found for m_b and taking now $\mu = 1/a$. However, since (as stated) radiative and nonperturbative corrections are large, the results are very sensitive to the value of μ chosen. For this reason it is more profitable to *fit* μ . This is the procedure followed in ref. 14, from where the following table for the mass splittings of the states shown is taken:

States	Theory	Experiment
$2^3P_2 - 2^3P_1$	21 ± 7	$21 \pm 1 \text{ MeV}$
$2^3P_1 - 2^3P_0$	29 ± 9	$32 \pm 2 \text{ MeV}$
$2^3S_1 - \overline{2^3P}$	181 ± 60	$123 \pm 1 \text{ MeV}$
$2^3S_1 - 1^3S_1$	428 ± 105	$563 \pm 0.4 \text{ MeV}$
$\overline{2^3P} - 2^1P_1$	1.5 ± 1	—

Here we use standard spectroscopic notation; the common, fitted value of μ is somewhat above 1 GeV, and the errors given are *only* those generated by the errors in Λ , $\langle\alpha_s G^2\rangle$ given above. The overall agreement of

theory and experiment is noteworthy, particularly considering that the value of the single free parameter, μ , is the *same* for all states. However, the situation is less satisfactory than what a cursory glance to the Table might suggest: both perturbative and nonperturbative corrections are large, and the results are somewhat unstable.

The *wave functions* for states with $n = 2$ present such large errors that the calculation using the methods described up till now become meaningless for them.

3.3. Spin-dependent shifts and leptonic decay rates

The evaluation of spin-dependent shifts, and decay rates follow patterns similar to those of the spin-independent energy shift evaluations. The expressions one finds are^[11,14],

$$M(V) - M(\eta) = m \frac{C_F^4 \alpha_s(\mu^2) \tilde{\alpha}_s(\mu^2)^3}{3} [1 + \delta_{\text{wf}} + \delta_{\text{NP}}]^2 \times \left\{ 1 + \left[\frac{\beta_0}{2} \left(\log \frac{a\mu}{2} - 1 \right) + \frac{21}{4} (\log C_F \tilde{\alpha}_s + 1) + B \right] \frac{\alpha_s}{\pi} + \frac{1161}{8704} \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 \tilde{\alpha}_s^6} \right\}; \quad (3.3a)$$

$$\Gamma(V \rightarrow e^+ e^-) = \Gamma^{(0)} \times [1 + \delta_{\text{wf}} + \delta_{\text{NP}}]^2 (1 + \delta_{\text{rad}}), \quad \Gamma^{(0)} = 2 \left[\frac{Q_b \alpha_{\text{QED}}}{M(V)} \right]^2 (m C_F \tilde{\alpha}_s(\mu^2))^3; \quad (3.3b)$$

$$\delta_{\text{rad}} = -\frac{4C_F \alpha_s}{\pi}; \quad \delta_{\text{wf}} = \frac{3\beta_0}{4} \left(\log \frac{a\mu}{2} - \gamma_E \right) \frac{\alpha_s}{\pi};$$

$$\delta_{\text{NP}} = \frac{1}{2} \left[\frac{270459}{108800} + \frac{1838781}{2890000} \right] \frac{\pi \langle \alpha_s G^2 \rangle}{m^4 \tilde{\alpha}_s^6}.$$

Here $V = \Upsilon, J/\psi$. The corrections are fairly large, particularly the radiative correction^[20] δ_{rad} . Because of this the calculation is less reliable than what one would have expected for $\bar{b}b$, and fails completely for $\bar{c}c$. With the values of m_b found before, one has the numerical results,

$$M(\Upsilon) - M(\eta) = 53.3 \pm 5.3 (A) \pm 5.3 (\langle \alpha_s G^2 \rangle) \pm 10 (\mu^2 = 7.859 \pm 25\%) \quad (3.4)$$

and

$$\Gamma(\Upsilon \rightarrow e^+ e^-) = 1.143 \pm 0.11 (A) \pm 0.11 (\langle \alpha_s G^2 \rangle) \pm 0.24 (\mu^2 = 7.859 \pm 25\%). \quad (3.5)$$

Higher order NP corrections due to some higher dimensional operators are also known for the decay rate (see ref. 16). They would produce a shift in the decay rate of ~ 0.11 keV, smaller than the contribution of $\langle \alpha_s G^2 \rangle$ or the uncertainty caused by e.g. varying μ around μ_0 . We do *not* include either in the evaluation or the error estimate.

The calculated value for the decay is in reasonable agreement with the experimental figure,

$$\Gamma_{\text{exp.}}(\Upsilon \rightarrow e^+ e^-) = 1.320 \pm 0.04 \text{ keV}.$$

4. Heavy quarkonia at long distances. Connection between the long and short distance regimes

Here we consider bound states of heavy quarks at *long* distances. This certainly includes $\bar{c}c$ with $n > 1$ and $\bar{b}b$ with $n > 2$; $n = 1$ for the first and $n = 2$ for the second are somewhat marginal. As stated in the previous section, perturbative QCD supplemented with leading NP effects fails now; but, fortunately, and since the average velocity of bound states decreases with increasing n , we expect the dynamics to be governed by a potential: our task is to determine it. This has been considered by a number of people^[17,21,22]. Here we will follow the Dosch–Simonov method, in the version of ref. 17, which will allow us to establish connection with the short distance analysis of the previous section.

The potential, that we denote by $U(r)$, is expected to exhibit a number of features. First of all, it should behave as σr at long distances, as follows from e.g. the lattice calculations. Secondly, it should contain a coulombic piece, so we write

$$U(r) = -\frac{\kappa}{r} + U_{\text{NP}}(r), \quad (4.1)$$

and, at short distances, one identifies $\kappa = C_F \alpha_s + \text{radiative corrections}$.

To find this potential we consider the Green's function in terms of the Wilson loop, working directly in the nonrelativistic approximation, and for large time T : for a $\bar{q}q$ pair:

$$\begin{aligned} G(x, \bar{x}; y, \bar{y}) &= \int \mathcal{D}z \mathcal{D}\bar{z} e^{-(K_0 + \bar{K}_0)} \langle W(C) \rangle, \\ \langle W(C) \rangle &= \int \mathcal{D}B e^{i \int_0^T dt (L_{\text{int}} + L_{\text{rad}})}. \end{aligned} \quad (4.2)$$

Note that we treat the quarks in the nonrelativistic quantum mechanical formalism, appropriate because of their nonrelativistic motion. Thus, K_0, \bar{K}_0 are time integrals of the kinetic energies (nonrelativistic lagrangians) of quark and antiquark,

$$K_0 = \frac{m}{2} \int_0^T d\mathbf{z}(t)^2, \quad \bar{K}_0 = \frac{m}{2} \int_0^T d\bar{\mathbf{z}}(t)^2.$$

However, the gluons are treated fully field-theoretically. So the radiation lagrangian is $L_{\text{rad}} = -\frac{1}{4} \int d^3r G^2$. The Wilson loop operator corresponds to the contour C enclosing the q, \bar{q} paths from time 0 to time T . It should include path-ordered parallel transporters for the initial and final states, $\Phi(x, \bar{x}), \Phi(y, \bar{y})$ with e.g. in matrix notation

$$\Phi(x, \bar{x}) = \text{P exp } ig \int_{\bar{x}}^x dz_\mu t_a B_\mu^a(z)$$

which we do not write explicitly.

To take into account the nonperturbative character of the interaction it is convenient to work in the background gauge formalism and write $B_\mu = b_\mu + a_\mu$ where the a_μ represent the quantum fluctuations and b_μ is a background field. This is constructed such that the vacuum expectation value of the Wick ordered products of the a_μ , and of the mixed a_μ, b_μ products vanish. Therefore, we may express the gluon correlator in terms of b_μ only:

$$\begin{aligned} \langle : G(x) G(y) : \rangle &\rightarrow \langle : G_b(x) G_b(y) : \rangle, \\ G_{b,\mu\nu} &= \partial_\mu b_\nu - \partial_\nu b_\mu + g b_\mu \times b_\nu. \end{aligned}$$

Expanding in powers of the background field b_μ we may write the Wilson loop average as

$$\begin{aligned} \langle W(C) \rangle &= \int \mathcal{D}a \text{Pe}^{\int_C dz_\mu a_\mu} \\ &+ \left(\frac{ig}{2!} \right)^2 \int \mathcal{D}a \int_C dz_\mu \int dz'_\nu \text{P}\Phi_a(z, z') b_\mu(z) \text{P}\Phi_a(z', z) b_\nu(z') + \dots \\ &\equiv W_0 + W_2 + \dots \end{aligned} \quad (4.3)$$

and the transporter Φ_a is constructed with only the quantum field a . For the first term, W_0 , the cluster expansion gives

$$\begin{aligned} W_0 &= Z \exp(\varphi_2 + \text{higher orders}), \\ \varphi_2 &= \frac{C_F g^2}{4\pi^2} \int_0^T dt \int_0^T dt' \frac{1 + \dot{\mathbf{z}}\dot{\mathbf{z}}'}{\mathbf{r}^2 + (t - t')^2} \\ &= C_F \alpha_s r^{-1} \int_0^T dt + O(v^2), \end{aligned}$$

i.e., the coulombic piece of the potential. (Z is a constant that, in particular, includes regularization).

We keep now only the next piece, W_2 and neglect the $W_n, n > 2$. At *short* distances, this is justified because higher W_n involve higher powers of the background fields. Thus the ensuing corrections

are suppressed, on dimensional grounds, by powers of $1/m^n$. For long distances, and although arguments have been advanced for the dominance of the W_2 , we really have a model, the so-called “stochastic vacuum model”.

For details of the evaluation of this first nontrivial piece, W_2 , we refer to the lectures by Simonov.³ It produces a correction to the Green’s function, δG , which in the static approximation is

$$\delta G = -\frac{1}{24} \int d^3r \int d^3r' \int r_i d\beta \int r'_i d\beta' \\ \times G_C^{(S)}(r(T), r) G_C^{(8)}(r, r') G_C^{(S)}(r', r(0)).$$

$G_C^{(S,8)}$ are the singlet, octet coulombic Green’s functions. The reason for the appearance of $G^{(8)}$ are similar to those for the appearance of $H^{(8)}$ in (2.9).

We may then take matrix elements between coulombic states, $|nl\rangle$, and identify the ensuing energy shifts from the relation

$$G = G_C^{(S)} + \delta G \underset{T \rightarrow \infty}{\simeq} G_C^{(S)}(1 - T\delta E_{nl}).$$

We then find the basic equation^[17],

$$\delta E_{nl} = \frac{1}{16} \int \frac{d^3p dp_0}{(2\pi)^4} \int d\beta d\beta' \tilde{\Delta}(p) \sum \langle nl | r_i e^{i\mathbf{p}(\beta-1/2)\mathbf{r}} | k(8) \rangle \\ \times \frac{1}{E_k^{(8)} - E_n - p_0} \langle k(8) | r'_i e^{i\mathbf{p}'(\beta-1/2)\mathbf{r}'} | nl \rangle. \quad (4.4)$$

The states $|k(8)\rangle$ are eigenstates of the octet Hamiltonian, with energy $E_k^{(8)}$; the E_n are the coulombic energies.⁴ Finally, $\tilde{\Delta}(p)$ is defined in terms of the correlators, being the Fourier transform of

$$\Delta(x) = D(x) + D_1(x) + x^2 \partial^2 D_1(x) / \partial x^2$$

and

$$\langle g^2 : G_{0i}(x) G_{0j}(0) : \rangle = \frac{1}{12} \left[\delta_{ij} D(x) + x_i x_j \frac{\partial^2 D_1}{\partial x^2} \right].$$

We may write, using Lorentz invariance, $\Delta(x) = \Delta(x^2/T_g^2)$, with T_g the so-called correlation time. This will play an important role in what follows.

We have now two regimes. If $\mu_T \equiv T_g^{-1} \gg |E_n|$ the velocity tends to zero, and the nonlocality also tends to zero as compared with the quark rotation period (which in the coulombic approximation would be $1/|E_n|$). We can now neglect, in Eq. (4.4), both E_n , $E_k^{(8)}$ as compared to p_0 so, after some elaboration, we obtain the energy shifts

$$\delta E_{nl} \simeq \langle nl | U_{NP} | nl \rangle$$

with

$$U_{NP}(r) = \frac{2r}{36} \left\{ \int_0^r d\lambda \int_0^\infty d\nu D(\lambda, \nu) \right. \\ \left. + \frac{1}{2} \int_0^r d\lambda^2 \int_0^\infty d\nu [-2D(\lambda, \nu) + D_1(\lambda, \nu)] \right\}, \quad (4.5)$$

$$D(\lambda, \nu) \equiv D(x_0^2, \mathbf{x}^2), \text{ etc.}$$

At large r , and as this equation shows, we find $U_{NP}(r) \simeq \sigma r$. Here σ can be related to T_g and the gluon condensate if we assume a model for Δ . So, if e.g. we take an exponential ansatz for $\Delta(x)$, as in ref. 22, we find

$$\mu_T = \frac{\pi}{3\sqrt{2}} \frac{\langle \alpha_s : G^2 : \rangle}{\sigma^{\frac{1}{2}}} \simeq 0.32 \text{ GeV}.$$

³ For technical reason, should be performed in euclidean space returning to Minkowski space at the end of the calculation. We omit these subtleties here.

⁴ Here we subtract the rest energy, $2m$, from both $E^{(8)}$, E_n .

For small r the limit of the expression in Eq. (4.5) leads to^[21],

$$U_{\text{NP}}(r) \simeq c_0 + c_1 r^2. \quad (4.6)$$

This is *different* from the behaviour expected from the Leutwyler–Voloshin analysis which gave a behaviour $\sim r^3$; but one should understand that the present derivation holds for $r \rightarrow 0$ but still $T_g^{-1} \gg |E_n|$, i.e., in a situation other than that where the Leutwyler–Voloshin analysis is valid.

It may be stated that the analysis based upon the potential (4.5) gives a very good description of heavy quarkonia states^[23]. Note, however, that the description is not the only one available on the market; others, based e.g. on relativistic corrections to an assumed linear potential are given in the papers in ref. 24.

We next get the matching between the two regimes^[17]. For this we turn to the opposite situation, viz., $T_g^{-1} \ll |E_n|$. Now we may approximate $\Delta(x) \sim \text{constant}$, so that $\tilde{\Delta}(p) \sim \delta_4(p)$ and Eq. (4.4) becomes

$$\delta_{\text{NP}} E_{nl} = \frac{\pi \langle \alpha_s : G^2 : \rangle}{18} \langle nl | r_i \frac{1}{H^{(8)} - E_n + \mu_T} r_i | nl \rangle, \quad (4.7)$$

which coincides exactly with the results of the Leutwyler–Voloshin analysis^[2] in the limit $T_g \rightarrow \infty$ ($\mu_T \rightarrow 0$): cf. Eq. (2.9). In fact, Eq. (4.7) allows us to estimate the finite size corrections to the Leutwyler–Voloshin NP effects, which improves still the agreement between theory and experiment^[17].

5. Further discussion of nonperturbative effects: Renormalons, and saturation

In the previous sections we have shown how QCD can give a very satisfactory account of the heavy quarkonia spectra, particularly of the lowest lying states; an understanding based on perturbative calculations supplemented by NP ones, in particular those associated with the gluon condensate. Here we address some questions related to that.

First, one may inquire about the connection of *renormalons* with nonperturbative effects. We return to the one-gluon exchange diagram, Fig. 2. If we dress the gluon propagator with loops as in Fig. 5 then the corresponding potential, in momentum space, is

$$\tilde{V}(k) = \frac{-4\pi C_F}{k^2} \frac{4\pi}{\beta_0 \log(k^2/\Lambda^2)}, \quad (5.1)$$

and we have substituted the one-loop expression for $\alpha_s(k^2)$. The expression (5.1) is undefined for *soft* gluons, with $k^2 \simeq \Lambda^2$. As follows from the general theory of singular functions, the ambiguity is of the form $c\delta(k^2 - \Lambda^2)$: upon Fourier transformation this produces an ambiguity in the x -space potential of $\delta V(r) = c[\sin \Lambda r]/r$. At short distances we may expand this in powers of r and find^[25,26]

$$\delta V(r) \sim C_0 + C_1 r^2 + \dots. \quad (5.2)$$

The same result may be obtained with the more traditional method of Borel transforms. The behaviour in Eq. (5.2) coincides with the short distance behaviour of the nonperturbative potential $U_{\text{NP}}(r)$ as determined in ref. 21, and Eq. (4.6) here.

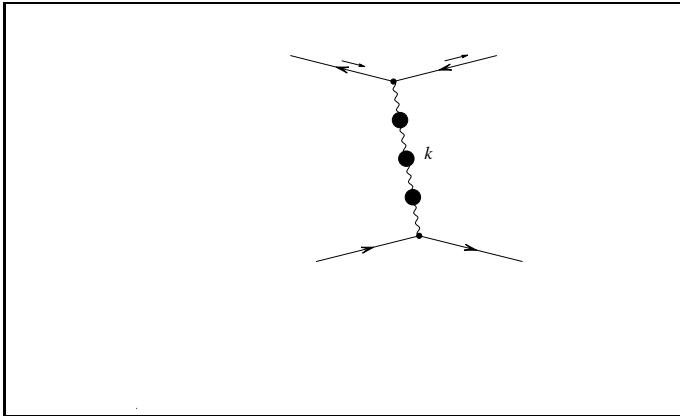


FIGURE 5. One-gluon exchange, dressed with loops.

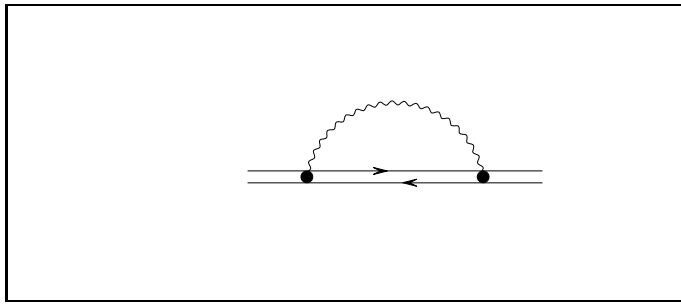


FIGURE 6. Emission and absorption of a soft gluon collectively by a $\bar{q}q$ pair.

The situation just described applies for states $\bar{q}q$ at short distances; but not so short that zero frequency gluons cannot separate the bound state. If this last is the case, soft gluons do not resolve the $\bar{q}q$ pair and only see a dipole. The basic diagram is no more that of Figs. 2, 5, but that of Fig. 6. The generated renormalon may then be seen^[26] to correspond to the contribution of the gluon condensate in the Leutwyler-Voloshin mechanism.

The matter of renormalons does not end here. If we calculate the renormalization of the mass to one loop, and dress the propagator with bubbles, one also finds a renormalon contribution to an ambiguity in the mass of^[27]

$$\delta_{\text{renormalon}} m = \Lambda^2/m,$$

and there is another nonperturbative correction to the hamiltonian related to the arbitrariness in the origin of the energies. Indeed, we fix this origin by requiring the potential to vanish at infinity but, since the quarks are confined in a region of radius $R \sim \Lambda^{-1}$, “infinity” is equivalent to R , hence we get an ambiguity of order $\Lambda \sim 1/R$.

The situation, however, is less confused than what one might think. In fact, and at least in the case in which $m|v|$ is large compared to Λ , it can be shown that the linear and quadratic renormalons in the potential and pole mass cancel, leaving a $\Lambda^4 r^3$ renormalon. This was to be expected on general grounds (see e.g., ref. 26); a formal proof may be found in ref. 28.

To make matters worse (or to improve them, depending on the viewpoint) we will also consider the possibility of *saturation*. We note that the ambiguities we have found are associated with small momenta or, equivalently, long distances. However, at least the *singularities* are clearly spurious. Indeed, not only the theory should be well defined but, because of confinement, long distances are never attained: the theory possesses an internal infrared cut-off of the order of the confinement radius, $R \sim \Lambda^{-1}$. To try and implement it we consider again the gluon propagator. To one loop it gets a correction involving the vacuum polarization tensor. Neglecting quarks this is, in x -space, given by an expression like

$$\Pi_{\alpha\beta}^{aa'}(x, 0) \sim g^2 f_{abc} f_{a'de} \langle 0 | \int d^4 y_1 d^4 y_2 \text{Tr} B_b^\alpha(y_1) \partial_\mu B_{c\alpha}(y_1) B_d^\beta(y_2) \partial_\nu B_{e\beta}(y_2) | 0 \rangle + \dots$$

We can take into account the *long distance* interactions by introducing a string between the field products at finite distances. In matrix notation for the gluonic fields, $\mathcal{B}^\mu = t_a B_a^\mu$, this is implemented by replacing

$$\mathcal{B}^\alpha(y_1) \mathcal{B}^\beta(y_2) \rightarrow \mathcal{B}^\alpha(y_1) \text{P} \left(\exp i \int_{y_2}^{y_1} dz^\mu \mathcal{B}_\mu(z) \right) \mathcal{B}^\beta(y_2).$$

The process may be described as “filling the loop” (see Fig. 7) by introducing all exchanges between the gluonic lines there. If we furthermore replace the perturbative vacuum $|0\rangle$ by the nonperturbative one $|\text{vac}\rangle$, then a calculation similar to that made for the long distance potential for heavy quarks in Sect. 4 yields a dressed propagator

$$D_{\text{dressed}}^{\mu\nu}(k) = D^{(0)\mu\nu}(k) \frac{4\pi}{\beta_0 \log(M^2 + k^2)/\Lambda^2},$$

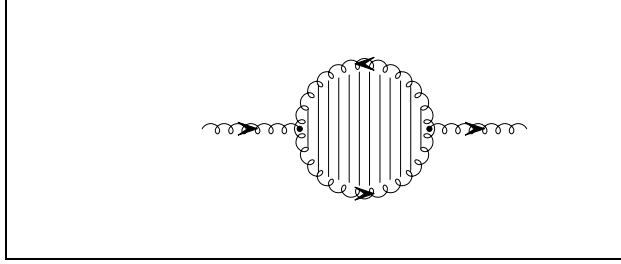


FIGURE 7. “Filled in” gluon loop.

and M^2 is related to the gluon condensate at finite distances, $\langle G(x)G(0) \rangle_{\text{vac}}$.

This indicates a *saturation* property of the coupling constant at small momenta (long distances); the calculation in fact suggests that, at small momenta, the expression for the running coupling constant should be modified according to

$$\alpha_s(k^2) = \frac{4\pi}{\beta_0 \log k^2/\Lambda^2} \rightarrow \alpha_s^{\text{sat}}(k^2) = \frac{4\pi}{\beta_0 \log(k^2 + M^2)/\Lambda^2}. \quad (5.3)$$

It is certain that an expression such as (5.3) incorporates, to some extent, long distance properties of the QCD interaction. For example, if we take (5.3) with $M = \Lambda$ in the tree level potential for heavy quarks, this becomes the *Richardson potential*^[29]

$$\tilde{V}^{(0)}(\mathbf{k}) = -\frac{4\pi C_F \alpha_s(\mathbf{k}^2)}{\mathbf{k}^2} \rightarrow \tilde{V}^{(0),\text{sat}}(\mathbf{k}) = -\frac{16\pi^2 C_F}{\beta_0 \mathbf{k}^2 \log(k^2 + \Lambda^2)/\Lambda^2}.$$

When one has $\mathbf{k}^2 \gg \Lambda^2$, the short distance coulombic potential is, of course, recovered. For $\mathbf{k}^2 \ll \Lambda^2$, however,

$$\tilde{V}^{(0),\text{sat}}(\mathbf{k}) \underset{\mathbf{k}^2 \ll \Lambda^2}{\simeq} \frac{16\pi^2 C_F \Lambda^2}{\beta_0 \mathbf{k}^4},$$

whose Fourier transform gives

$$V^{(0),\text{sat}}(r) \underset{r \gg \Lambda^{-1}}{\simeq} (\text{constant}) \times r,$$

i.e., a linear potential. Indeed, a reasonably accurate description of spin-independent splittings in quarkonia states is obtained with such a potential. Likewise, use of (5.3) with $M = \Lambda$ provides a surprisingly good description of small- x deep inelastic scattering down to $Q^2 \sim 0$, as discussed in ref. 29; and these two cases are not unique.

In spite of these successes, it should nevertheless be obvious that (5.3) can only be of limited applicability. For example, consider the correlator of two currents in the spacelike region, $\Pi(Q^2)$. We know that in some cases such as the correlators of vector or axial currents for massless quarks, or that of pseudoscalar ones, one has

$$\Pi(Q^2) \underset{Q^2 \rightarrow \infty}{\simeq} \Pi_{\text{perturbative}} \{1 + O(\langle \alpha_s G^2 \rangle) Q^{-4}\},$$

whereas (5.3) would give a correction of order $M^2 Q^{-2}$. The Richardson potential is also a good example of the limitations of the uses of saturation, in particular in connection with the extent to which saturation really does (or does not) represent a real, physical improvement, or merely the addition of a somewhat arbitrary new parameter. Indeed, the linear potential induced by saturation in the Richardson model is the *fourth component* of a Lorentz vector, while we know that the Wilson linear potential, as obtained, e. g., in the stochastic vacuum model or in lattice calculations, should be a Lorentz four-scalar: it thus follows that the linear potential obtained from saturation can be only of phenomenological use in some specific situations.

It is not easy to draw a clear morale from all of this. One can try to eschew the problem by expressing observables in terms of observables (for example, $\Gamma(\mathcal{T} \rightarrow e^+ e^-)$ in terms of $M(\mathcal{T})$), hoping that this will reduce renormalon ambiguities, as some calculations seem to indicate^[27,30]. This is the viewpoint adopted in the papers in ref. 31. Another possible attitude is the following. It is very likely that the perturbative series in QCD are divergent; hence, different methods of summation lead to different results. This appears to be the case for renormalons or saturation resummations. It is the author's belief that only if the summation method is rooted on solid physics it is likely to represent an improvement; otherwise, estimates of nonperturbative

effects become pure guesswork. In this respect, the method of taking into account the nonperturbative nature of the physical vacuum by considering the effect of nonzero values for the correlators stands some chance of being meaningful, as indeed phenomenological calculations seem to indicate.

6. Models

6.1. The Constituent Quark Model

We first discuss the constituent quark model. Here, we assume that the fact that quarks inside hadrons move through a medium made up of gluons and quark-antiquark pairs can, under certain circumstances, be represented by ascribing an effective mass, called the *constituent* mass, even to light quarks. A possible way to connect this with a QCD analysis might be the following.⁵ Consider the quark propagator, in the physical, nonperturbative vacuum that we denote by $|\text{vac}\rangle$,

$$S_{ij}(x) = \langle \text{vac} | T q_i(x) \bar{q}_j(0) | \text{vac} \rangle;$$

it is a gauge dependent object. We may define an invariant propagator by inserting a line integral. In matrix notation, but working in Minkowski space for now, we thus write an effective propagator as

$$S_{ij}^{\text{eff}}(x) = -\frac{\delta_{ij}}{N_c} \langle \text{vac} | T \bar{q}(0) P \exp^{-ig \int_x^0 dy^\mu B_\mu(y)} q(x) | \text{vac} \rangle. \quad (6.1)$$

We can interpret S_{ij}^{eff} as the propagator describing a quark as it moves in the gluonic soup inside a hadron. In p -space,

$$S_{ij}^{\text{eff}}(p) = \int d^4x e^{ip \cdot x} S_{ij}^{\text{eff}}(x).$$

Evaluating the short distance limit with the OPE, we get the familiar lowest order expression

$$S_{ij}^{\text{eff}}(x) \underset{x \rightarrow 0}{\simeq} \delta_{ij} \left\{ \frac{-1}{4\pi^2} \not{\partial} \frac{1}{x^2 - i0} - \frac{1}{4N_c} \langle \bar{q}q \rangle \right\}; \quad (6.2)$$

for simplicity we have taken the quark to be massless. At long distances we evaluate $S_{ij}^{\text{eff}}(x)$ as follows. First, we go to Euclidean space. Then, and because we expect confinement (and thus that the interaction grows at long distances), we calculate for large coupling, $g \rightarrow \infty$. Finally, the quenched approximation is used.

Under these circumstances, the evaluation of $S_{ij}^{\text{eff}}(x)$ is identical to that of the Wilson loop. Underlining Euclidean quantities, we then find

$$S_{ij}^{\text{eff}}(\underline{x}) \underset{\underline{x} \rightarrow \infty}{\sim} \delta_{ij} e^{-\sigma^{1/2} |\underline{x}|};$$

σ is the string tension. In Minkowski space this becomes

$$S_{ij}^{\text{eff}}(x) \underset{\underline{x} \rightarrow \infty}{\sim} \delta_{ij} e^{-\sigma^{1/2} \sqrt{-x^2}}, \quad (6.3)$$

an expression which is very appealing. According to it, the probability of a quark in the vacuum (inside a hadron) to propagate at a spacelike distance $r = \sqrt{-x^2}$ decreases exponentially when $r \gg \sigma^{-1/2}$; but the quark may move freely along a timelike or lightlike trajectory, where $\sqrt{-x^2}$ is pure imaginary or zero.

A simple ansatz incorporating short and long distance behaviour is

$$S_{ij}^{\text{eff}}(x) = \delta_{ij} \left\{ \frac{-1}{4\pi^2} \not{\partial} \frac{1}{x^2 - i0} - \frac{1}{4N_c} \langle \bar{q}q \rangle \right\} e^{-\sigma^{1/2} \sqrt{-x^2}}. \quad (6.5a)$$

The corresponding p -space expression is then easily evaluated to be

$$S_{ij}^{\text{eff}}(p) = \delta_{ij} \left\{ \frac{i}{\not{p}} \left(1 - \frac{\sigma^{1/2}}{(\sigma - p^2 - i0)^{1/2}} \right) - \frac{3\pi^2 i \sigma^{1/2} \langle \bar{q}q \rangle}{N_c (K - p^2 - i0)^{5/2}} \right\}. \quad (6.5b)$$

⁵ Other mechanisms for the generation/interpretation of a constituent mass may be found in the lectures by Yu. Simonov, and in ref. 32.

The expression (6.5) for the propagator fulfills the Bricmont–Fröhlich^[33] criterion for confinement and indeed exhibits many of the characteristics of the propagator for a particle with nonzero effective mass. Thus, $S_{ij}^{\text{eff}}(p)$ presents a cut starting at $p^2 = K$ and, what is more interesting, it behaves for $p \rightarrow 0$ like the propagator for a massive particle:

$$S_{ij}^{\text{eff}}(p) \underset{p \rightarrow 0}{\simeq} -\frac{i\not{p}}{2\sigma} + \frac{3\pi^2 i \langle \bar{q}q \rangle}{N_c \sigma^2} \simeq -\frac{i}{\mu_0}, \quad (6.6)$$

where the effective mass μ_0 is

$$\mu_0 = \frac{N_c \sigma^2}{-3\pi^2 \langle \bar{q}q \rangle}. \quad (6.6)$$

It is curious that in the last expression the quark condensate appears in the denominator.

The numerology works reasonably well. With the value $\mu_0 \simeq 320$ MeV obtained from phenomenological quark models, we can predict $\sigma^{1/2} \simeq 470$ MeV, in reasonable agreement with lattice calculation results that give $\sigma^{1/2} \simeq 420$ MeV. These nice features should, however, not make one forget the shortcomings of our calculation here; (6.5) is to be considered no more than a phenomenological expression. In fact, not only is the interpolation used somewhat arbitrary, but, because the expression for the propagator only takes account of a certain class of gluon couplings, use of (6.5) into Feynman diagrams may lead to violations of gauge invariance. Because of this it is probably better not to ask too much of the model and take, simply, the consequence of the existence of a universal mass that represents the inertia acquired by quarks due to their having to drag in their motion the gluon–quark soup, and which adds to mechanical quark masses. Concentrating on light quarks, we then assume masses

$$m_u(\text{const}) = m_u + \mu_0, \quad m_d(\text{const}) = m_d + \mu_0, \quad m_s(\text{const}) = m_s + \mu_0. \quad (6.7)$$

The presence of the mass μ_0 breaks chiral invariance, and therefore pions and kaons (in particular) will be very poorly described by the constituent quark model: for these particles we have to use different methods. But one can use the constituent quark model to describe with success other hadrons (ρ , K^* , Σ , Λ , nucleons, Δ , \dots).

To implement the interactions among quarks, we introduce two phenomenological potentials: a confining potential, linear in r ,

$$U_{\text{conf}}(r) = \lambda r, \quad \lambda \sim \sigma, \quad (6.8a)$$

and a coulombic-type interaction,

$$U_{\text{Coul}}(r) = \frac{-\kappa}{r}, \quad (6.8b)$$

together with corresponding QCD-type hyperfine interactions. For quarks with indices i, j , we take

$$U_{\text{hyp}}(r) = -\kappa \sum_{i \neq j} \frac{1}{m_i m_j} \sum_a t_i^a t_j^a \boldsymbol{\sigma}_i \boldsymbol{\sigma}_j, \quad (6.8c)$$

and t_i , $\boldsymbol{\sigma}_i$ act on the wave function of quark i . κ may be connected with the running coupling at, say, the reference momentum of 1 GeV:

$$\kappa \sim C_F \alpha_s(1 \text{ GeV}^2).$$

Because the model is in any case not terribly precise, one at times replaces the linear potential by a quadratic potential, which can be solved explicitly.

6.2. The Bag Model

In the bag model one neglects confinement, and replaces it by the boundary condition that quarks cannot leave a “bag” with size $R \sim \Lambda^{-1}$. There are a number of variants of the bag: the quantum mechanical bag, the field theoretic bag, little bags, etc.^[34,35] The simplest, of course, is the quantum mechanical bag^[34]. There, as a first approximation, one considers quarks as free, subject only to the boundary condition that the wave function vanish for $r = R$. This bag is solved by writing the Dirac equation for free particles and imposing then the boundary condition. The simple model produces *qualitatively* reasonable results for light quark states, with the exception of pionic and kaonic states: as was to be expected, because the presence of the bag breaks chiral invariance. It is also not clear to what extent the presence of the bag is a good simulation of the confinement mechanisms or, more generally, nonperturbative effects: in fact it is *not* for heavy quarkonia. To see this, consider a system of heavy quarks, $\bar{q}q$, inside a spherical bag of radius R . We take the interaction to be the coulombic one, $-C_F\alpha_s/r$, and impose the bag boundary condition.

Let us denote by m_r to the reduced quark mass, $m_r = m/2$ for quarkonium. For a state with energy E , define $n_E \equiv \sqrt{\text{Ry}/(-E)}$, and the variable $\rho = 2r/n_E a$, where $a = 1/m_r C_F\alpha_s$ and $\text{Ry} = \frac{1}{2}m_r(C_F\alpha_s)^2$. For states with $l = 0$, the differential equation obeyed by the radial wave function Ψ_E is

$$P''(\rho) + \left(\frac{2}{\rho} - 1\right) P'(\rho) + \frac{n_E - 1}{\rho} P(\rho) = 0,$$

$$\Psi_E(\rho) = (\text{Const.}) \times e^{-\rho/2} P(\rho).$$

Moreover we have the bag boundary condition $P(L) = 0$, where $L = 2R/n_E a$.

We are interested in the solution of this for $L \gg a$. To find it, we proceed as follows. The regular solution is proportional to Kummer’s function, $P(\rho) = M(1 - n_E, 2; \rho)$; the boundary condition then fixes n_E . To see this, consider the ground state. Since in the limit $L \rightarrow \infty$ we should recover the ordinary solution, with $n_E = 1$, we write $n_E = 1 + \epsilon$ and work to lowest order in ϵ . Expanding,

$$\begin{aligned} M(1 - n_E, 2; \rho) &= M(-\epsilon, 2; \rho) \simeq M(0, 2; \rho) + \epsilon \frac{\partial}{\partial a} M(a, 2; \rho) \Big|_{a=0} \\ &= 1 + \frac{1}{\Gamma(-\epsilon)} \sum_{n=1}^{\infty} \frac{\Gamma(n)}{n! \Gamma(n+2)} \rho^n \simeq 1 - \epsilon \eta(\rho), \\ \eta(\rho) &\equiv \sum_{n=1}^{\infty} \frac{\rho^n}{n(n+1)!}. \end{aligned}$$

For this to have a zero at $\rho = L$ we must have

$$\epsilon^{-1} = \eta(L) = \sum_{n=1}^{\infty} \frac{L^n}{n(n+1)!}.$$

At large L , $\eta(L) \underset{L \rightarrow \infty}{\simeq} e^L/L^2$, hence $\epsilon \simeq L^2 e^{-L} = (2R/a)^2 e^{-2R/a}$. For the energy, therefore,

$$E = -\text{Ry} + 2m_r^3 R^2 (C_F\alpha_s)^2 e^{-2R/a}.$$

In the case of quarkonium, the nonperturbative shift induced in the ground state by the presence of the bag of radius R is thus

$$\delta_{\text{NP}} E = \frac{1}{4} m^3 R^2 (C_F\alpha_s)^2 e^{-RmC_F\alpha_s},$$

totally different from what is found both at short distances (as caused by the gluon condensate, Eq. (2.10)) or at long distances, as given by a linear potential σr that yields something proportional to $\sigma/m\alpha_s$.

Appendix: Constants

We collect here, for ease of reference, some of the constants that appear in the radiative corrections to heavy quarkonium states.

$$\begin{aligned}\beta_0 &= 11 - \frac{2}{3}n_f; \beta_1 = 102 - \frac{38}{3}n_f \\ \beta_2 &= \frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2 \\ a_1 &= \frac{31C_A - 20T_F n_f}{36} \simeq 1.47; \quad b_1 = \frac{C_F - 2C_A}{2} \simeq -2.33; \\ a_2 &= \frac{1}{16} \left\{ \left[\frac{4343}{162} + 4\pi^2 - \frac{1}{4}\pi^4 + \frac{22}{3}\zeta(3) \right] C_A^2 \right. \\ &\quad \left. - \left[\frac{1798}{81} + \frac{56}{3}\zeta(3) \right] C_A T_F n_f - \left[\frac{55}{3} - 16\zeta(3) \right] C_F T_F n_f + \frac{400}{81} T_F^2 n_f^2 \right\}; \\ c_2^{(L)} &= a_1 \beta_0 + \frac{1}{8} \beta_1 + \frac{1}{2} \gamma_E \beta_0^2.\end{aligned}$$

$$B = \frac{3}{2}(1 - \log 2)T_F - \frac{5}{9}T_F n_f + \frac{11C_A - 9C_F}{18}.$$

$$\begin{aligned}N_1^{(n,l)} &= \frac{\psi(1+l+n)-1}{2}; \quad N_0^{(n,l)} = \frac{1}{4}\psi(1+l+n)[\psi(1+l+n)-2] \\ &\quad + \frac{n}{2} \left\{ \frac{(n-l-1)!}{(n+l)!} \sum_{s=0}^{n-l-2} \frac{(s+2l+1)!}{s!(s+l+1-n)^3} \right. \\ &\quad \left. + \frac{(n+l)!}{(n-l-1)!} \sum_{s=n-l}^{\infty} \frac{s!}{(s+2l+1)!(s+l+1-n)^3} \right\}.\end{aligned}$$

We remark that the definitions of a_2 and b_1 have been swapped, in contrast to refs. 11, 13, 14, but in agreement with refs. 10.

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$$\delta E_{10} = -m[C_F + \frac{3}{2}C_A]C_F^4\alpha_s^5(\log \alpha_s)/\pi$$

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